Straight-edge diffraction of Planck radiation

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The irradiance diffraction profile of a straight edge is given as a Taylor series in powers of the distance from the geometrical shadow boundary to any point in the profile for monochromatic radiation. The coefficients of the series, which are obtained as simple analytic expressions, are proportional to the real part of a complex number whose phase cycles through a complete period every eight terms in the series. Integration of this series over a Planck distribution of radiation yields the power series for the Planck profile; this derived series has a finite radius of convergence. The asymptotic series for the Planck profile far from the shadow boundary and beyond the radius of convergence of its power series is obtained by analytic continuation of the power series with the aid of a Barnes type of integral representation.

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I. INTRODUCTION

Diffraction destroys the sharpness of shadow boundaries defined by geometrical optics even in the case of point sources. When an edge is assumed to define where radiation lands downstream in an optical system, based on the projected geometrical shadow boundary, diffraction-induced spillover can affect radiometric measurements. The effects depend on the context, such as whether an edge is straight, is the perimeter of a circular aperture or lens, or is the edge of an obfuscating disk. Whatever the case, as long as the radius of curvature of the edge is large compared to the scale of the diffraction pattern, a straight edge can provide a reasonable approximation to the irradiance profile in the vicinity of the projected geometrical shadow boundary in the plane of the detector. The geometry for production of the straight-edge diffraction pattern is illustrated in Fig. 1.

For monochromatic radiation, the profile is described in scalar Fresnel diffraction theory by Fresnel integrals. For example, Born and Wolf discuss this in detail [1] and provide a series expansion for the amplitude of this profile as a function of the distance from the geometrical shadow boundary in the horizontal plane in Fig. 1. As far as we know, power series expansions have been given only for the Fresnel integrals themselves, which describe the wave field *amplitude*, but not for the *intensity* profile. The intensity profile is the square of the amplitude or, in the Planck radiation case, a weighted integral of the monochromatic intensity profile. In practice, not only monochromatic radiation, but also complex radiation, and Planck radiation in particular, is of interest. Therefore, this work provides the irradiance profile for Planck radiation diffracted by a straight edge in terms of

(1) The Taylor series in powers of the distance from the geometrical shadow boundary.

(2) The asymptotic series far from the geometrical shadow boundary.

These results are obtained by the following strategy. The power series for the monochromatic intensity distribution is obtained by writing a differential equation for the intensity as a function of the distance from the geometrical shadow boundary. The power-series solution of the differential equation is obtained. From this power series, the power series for the Planck radiation case is obtained by term-by-term integration of the monochromatic power series. Finally, the asymptotic expansion for the intensity in the Planck radiation case is obtained from its power-series expansion by employing a Barnes-type contour integral to carry out the analytic continuation [2].

The results reported here can be used to estimate the effects of diffraction of thermal radiation. For instance, these results can help determine the radiative heat load on a cryogenic detector that is some distance downstream from an optical element and some distance from the geometrical shadow boundary. Also, because Fresnel diffraction is a good approximation for edges with sufficiently large radii of curvature near the geometrical shadow boundary, these results exemplify the diffraction effects of the geometrical shadow boundary in many geometries, and suggest qualitative aspects of the diffraction effects for other similar broadband sources.

Since the pioneering works by Sanders and Jones [3], Ooba [4], and Blevin [5], accounting for diffraction effects has been an international effort in the most accurate radiometry. Most of the work has been devoted to assessing diffraction effects in the case of circular apertures [6], for which Boivin investigated diffraction effects near the aperture's geometrical shadow boundary [7]. Beyond Blevin's work, there has been some work on the diffraction of Planck radiation [8,9]. This is an especially important problem, as it relates to monitoring any changes in total solar irradiance, which may affect climate change. For example, diffraction analysis is often provided for various radiometers [10,11]. In Ref. [11], a 0.13% enhancement of the apparent value of total solar irradiance is expected from diffraction effects in a radiometer. This implies an error of about 1.8 $\mathrm{W/m^2}$ in the deduced solar radiance. Omitting an appropriate correction for diffraction would yield a measurement result that is not sufficiently accurate for the purpose of monitoring solar and related global climate variations.

II. STRAIGHT-EDGE FRESNEL DIFFRACTION OF MONOCHROMATIC LIGHT

In this section, the conventional theory of Fresnel diffraction by a straight edge based on the Kirchhoff theorem is reviewed to define notation and to provide the starting point for the calculations in this paper.

For a monochromatic scalar wave, with angular wave number k that originates at a point $\boldsymbol{r}''$, with coordinates x'', y'', and z'', the field $U(\boldsymbol{r})$ at a point $\boldsymbol{r}$, with coordinates x, y, and z, has the form

$$U(\boldsymbol{r}) = \frac{A\exp(ik|\boldsymbol{r}''-\boldsymbol{r}|)}{|\boldsymbol{r}''-\boldsymbol{r}|}, \tag{1}$$

where A is the amplitude of the spherical wave and is a property of the source. If the wave is partially blocked by a straight-edge half plane defined by $z=z'$, so that it can pass through all points with coordinates x, y, and z' in the plane with $x>0$, then the field downstream from the plane differs from the form described in Eq. (1).

In Kirchhoff's diffraction theory, in the paraxial approximation, the radiation field at a point $\boldsymbol{r}$ downstream from the plane is approximated as follows. With the shorthand $d_s = |z''-z'|$ and $d_d=|z'-z|$, Kirchhoff's theory gives

$$U(\boldsymbol{r}) = \frac{A}{i\lambda d_s d_d}\int_0^\infty dx' \int_{-\infty}^{\infty} dy' \exp[ikL(\boldsymbol{r}'',\boldsymbol{r}',\boldsymbol{r})], \tag{2}$$

where $\lambda=2\pi/k$, and the optical path length $L(\boldsymbol{r}'',\boldsymbol{r}',\boldsymbol{r})$ from $\boldsymbol{r}''$ to $\boldsymbol{r}'$ to $\boldsymbol{r}$ is approximated by

$$\begin{aligned} L(\boldsymbol{r}'',\boldsymbol{r}',\boldsymbol{r}) &= d_s + d_d + \frac{(x'-x'')^2+(y'-y'')^2}{2d_s} \\ &\quad + \frac{(x'-x)^2+(y'-y)^2}{2d_d} \\ &= \gamma_0 + \gamma_1 x' + \gamma_1' y' + \gamma_2(x'^2+y'^2), \end{aligned} \tag{3}$$

with

$$\gamma_0 = d_s + d_d + \frac{x''^2+y''^2}{2d_s} + \frac{x^2+y^2}{2d_d}, \tag{4}$$

$$\gamma_1 = -\frac{x''}{d_s} - \frac{x}{d_d}, \tag{5}$$

$$\gamma_1' = -\frac{y''}{d_s} - \frac{y}{d_d}, \tag{6}$$

$$\gamma_2 = \frac{1}{2d_s} + \frac{1}{2d_d}. \tag{7}$$

To avoid ambiguity in the integral in Eq. (2) and in subsequent integrals, a small positive imaginary part is included in k such that $\mathrm{Im}\,k=\epsilon$, and the limit $\epsilon\to 0$ is taken after the integral is evaluated. Without this replacement, the integral is not completely defined.

The expression for the intensity is normalized to the intensity U_0^2 that would result with no screen obscuring the radiation. That unobscured intensity is obtained from the absolute value squared of the integral in Eq. (2) with the lower limit for integration over x' replaced by $-\infty$. In this case, the completion of the squares in the exponent and the rewriting of the integration over x' and y' in polar coordinates yields

$$\begin{aligned} U_0^2 &= \left|\frac{2\pi A}{i\lambda d_s d_d}\int_0^\infty dr' r' \exp(ik\gamma_2 r'^2)\right|^2 \\ &= \frac{|A|^2}{4(d_s d_d \gamma_2)^2} = \frac{|A|^2}{(d_s+d_d)^2}. \end{aligned} \tag{8}$$

The spectral irradiance is thus given by

$$\begin{aligned} \frac{|U(\boldsymbol{r})|^2}{U_0^2} = \frac{4\gamma_2^2}{\lambda^2}\bigg|\int_0^\infty dx' \int_{-\infty}^{\infty} dy' \exp\{ik[\gamma_0 + \gamma_1 x' + \gamma_1' y' \\ + \gamma_2(x'^2+y'^2)]\}\bigg|^2. \end{aligned} \tag{9}$$

The integral over y' is evaluated by noting that

$$\begin{aligned} &\left|\int_{-\infty}^{\infty} dy' \exp[ik(\gamma_1' y' + \gamma_2 y'^2)]\right|^2 \\ &= \left|\int_{-\infty}^{\infty} dy' \exp(ik\gamma_2 y'^2)\right|^2 \\ &= \left|(k\gamma_2)^{-1/2}\int_0^\infty du\, u^{-1/2}\exp(-u)\right|^2 \\ &= \frac{\pi}{k\gamma_2}, \end{aligned} \tag{10}$$

so that

$$\begin{aligned} \frac{|U(\boldsymbol{r})|^2}{U_0^2} &= \frac{2\gamma_2}{\lambda}\left|\int_0^\infty dx' \exp[ik(\gamma_1 x' + \gamma_2 x'^2)]\right|^2 \\ &= \frac{2\gamma_2}{\lambda}\left|\int_{-x_0}^\infty dx' \exp(ik\gamma_2 x'^2)\right|^2, \end{aligned} \tag{11}$$

where

$$x_0 = -\frac{\gamma_1}{2\gamma_2}. \tag{12}$$

The change of variable to $t=(k\gamma_2)^{1/2}x'$ with $t_0 = (k\gamma_2)^{1/2}|x_0|$, where $\mathrm{Re}(k\gamma_2)^{1/2}>0$, gives the following result. In the illuminated region ($x_0>0$), one has

$$\frac{|U(\boldsymbol{r})|^2}{U_0^2} = \frac{1}{\pi}\left|\int_{-\infty}^{\infty} dt \exp(it^2) - \int_{t_0}^{\infty} dt\exp(it^2)\right|^2. \tag{13}$$

In the shadow region ($x_0<0$), one has

$$\frac{|U(\boldsymbol{r})|^2}{U_0^2} = \frac{1}{\pi}\left|\int_{t_0}^{\infty} dt \exp(it^2)\right|^2. \tag{14}$$

Both cases can be combined by writing

$$\frac{|U(\mathbf{r})|^2}{U_0^2} = \left| \theta(x_0) - \frac{1}{(i\pi)^{1/2}} \int_{t_0}^{\infty} dt \exp(it^2) \right|^2, \tag{15}$$

where

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0, \end{cases} \tag{16}$$

and $\mathrm{Re}(i\pi)^{1/2} > 0$ and t_0 is a positive real number. The integral in Eq. (15) is conventionally evaluated in terms of Fresnel integrals, as discussed by Born and Wolf [1]. The power series for the integral is simply

$$\int_{t_0}^{\infty} dt \exp(it^2) = \frac{(i\pi)^{1/2}}{2} - \sum_{n=0}^{\infty} \frac{i^n t_0^{2n+1}}{(2n+1)\Gamma(n+1)}. \tag{17}$$

Equation (15) immediately gives the result for the geometrical optics limit, i.e., $\lambda \to 0$, because the integral vanishes for $t_0 \to \infty$. This is confirmed by writing the variable t in terms of its real and imaginary parts $t = t_1 + it_2$, which gives $it^2 = i(t_1^2 - t_2^2) - 2t_1 t_2$. This form indicates that the path of integration can be extended to $|t| \to \infty$ anywhere in the first quadrant. Selecting the path in the positive t_2 direction beginning at t_0 gives

$$\left| \int_{t_0}^{t_0 + i\infty} dt \exp(it^2) \right| = \left| \int_0^{\infty} dt_2 \exp[i(t_0^2 - t_2^2) - 2t_0 t_2] \right| < \int_0^{\infty} dt_2 \exp(-2t_0 t_2) = \frac{1}{2t_0}. \tag{18}$$

III. DIFFRACTION EFFECTS NEAR THE SHADOW BOUNDARY

Near the shadow boundary, it is useful in the monochromatic case to expand $|U(\mathbf{r})/U_0|^2$ as a power series in the dimensionless parameter, $\tau = (k\gamma_2)^{1/2} x_0$. In this way, the coefficients of powers of τ are independent of the wavelength and details of the geometry.

From Eqs. (13) and (14), in either the illuminated region, $\tau > 0$, or the shadow region $\tau < 0$, the intensity is given by

$$\frac{|U(\mathbf{r})|^2}{U_0^2} = f(-\tau), \tag{19}$$

where the function f is defined as

$$f(\tau) = \frac{1}{\pi} \left| \int_{\tau}^{\infty} dt \exp(it^2) \right|^2 = \frac{1}{\pi} \int_{\tau}^{\infty} ds \exp(-is^2) \int_{\tau}^{\infty} dt \exp(it^2). \tag{20}$$

For this function

$$f(0) = \tfrac{1}{4} \tag{21}$$

and

$$f'(\tau) = \frac{d}{d\tau} f(\tau) = -\frac{\exp(-i\tau^2)}{\pi} \int_{\tau}^{\infty} dt \exp(it^2) - \frac{\exp(i\tau^2)}{\pi} \int_{\tau}^{\infty} dt \exp(-it^2). \tag{22}$$

A function g is defined here as

$$g(\tau) = \frac{\exp(-i\tau^2)}{\pi} \int_{\tau}^{\infty} dt \exp(it^2), \tag{23}$$

so that

$$f'(\tau) = -2 \,\mathrm{Re}\, g(\tau) \tag{24}$$

and

$$\frac{d}{d\tau} \exp(i\tau^2) g(\tau) = -\frac{\exp(i\tau^2)}{\pi}, \tag{25}$$

or

$$g'(\tau) + 2i\tau g(\tau) + \frac{1}{\pi} = 0, \tag{26}$$

with

$$g(0) = \left(\frac{i}{4\pi}\right)^{1/2}, \tag{27}$$

where $\mathrm{Re}\, g(0) > 0$. A trial power series for $g(\tau)$ is written as

$$g(\tau) = \sum_{n=0}^{\infty} a_n \tau^n, \tag{28}$$

which together with Eq. (27) and Eq. (26) with τ set to zero yields

$$a_0 = \left(\frac{i}{4\pi}\right)^{1/2}, \quad a_1 = -\frac{1}{\pi}. \tag{29}$$

The coefficients a_n for $n \geq 2$ are determined by substituting the series in Eq. (28) into Eq. (26) and equating like powers of τ, which gives

$$a_n = -\frac{2i}{n} a_{n-2}. \tag{30}$$

These coefficients yield the series solution given by

$$g(\tau) = \left(\frac{i}{4\pi}\right)^{1/2} \sum_{l=0}^{\infty} \frac{(\tau e^{3\pi i/4})^l}{\Gamma(1 + l/2)}, \tag{31}$$

and hence

$$f'(\tau) = -\mathrm{Re}\left[\left(\frac{i}{\pi}\right)^{1/2} \sum_{l=0}^{\infty} \frac{(\tau e^{3\pi i/4})^l}{\Gamma(1 + l/2)}\right]. \tag{32}$$

Integration over τ yields

$$f(\tau) = \frac{1}{4} + \mathrm{Re}\left[\frac{i}{(\pi)^{1/2}}\sum_{l=1}^{\infty}\frac{(\tau e^{3\pi i/4})^l}{l\Gamma(1/2+l/2)}\right]. \tag{33}$$

Thus, from Eq. (19), we have

$$\frac{|U(\boldsymbol{r})|^2}{U_0^2} = \frac{1}{4} + \mathrm{Im}\left[\frac{1}{(\pi)^{1/2}}\sum_{l=1}^{\infty}\frac{(\tau e^{\pi i/4})^l}{l\Gamma(1/2+l/2)}\right] = \sum_{l=0}^{\infty}\frac{\Gamma(1/2)\sin(l\pi/4)}{\Gamma(1/2+l/2)l\pi}\tau^l, \tag{34}$$

where in the last line, the term with $l=0$ is understood to be the limit as l approaches zero of the general term. Evidently, this power series converges for any value of τ. Consequences of the factor $\sin(l\pi/4)$ are that all coefficients of τ^l, where l is a nonzero multiple of four are zero, the other coefficients alternate signs in groups of three such that three are positive, three are negative, etc., and the terms with odd l include a factor of $\sqrt{2}$.

IV. PLANCK DISTRIBUTION SOURCE

A thermal source that emits a Planck distribution of radiation differs from the monochromatic source at $\boldsymbol{r}''$ that was assumed earlier. A thermal source is invariably extended, and its radiance depends on the source temperature T. The resulting diffraction profile can be obtained from the foregoing analysis if one considers a small element of the source area dA'' near $\boldsymbol{r}''$ and the contribution to the irradiance per unit source area, $dE(\boldsymbol{r},T)/dA''$. The analogous quantity in the absence of the diffracting edge may be denoted by $dE_0(\boldsymbol{r},T)/dA''$. The ratio of these two quantities, $\langle F(T)\rangle = [dE(\boldsymbol{r},T)/dA'']/[dE_0(\boldsymbol{r},T)/dA'']$, can be obtained from the above power series using Planck's law. This ratio is given by

$$\langle F(T)\rangle = \left[\int_0^{\infty} dk \frac{k^3}{e^{\beta k}-1}\frac{|U(\boldsymbol{r})|^2}{U_0^2}\right] \Bigg/ \left[\int_0^{\infty} dk' \frac{k'^3}{e^{\beta k'}-1}\right], \tag{35}$$

where $\beta = c_2/(2\pi T)$, and the dependence on $\boldsymbol{r}$ enters through the variable $\tau = (k\gamma_2)^{1/2}x_0$. The identity

$$\int_0^{\infty} dk \frac{k^3\tau^l}{(e^{k\beta}-1)} = \beta^{-4}\Gamma(4+l/2)\zeta(4+l/2)u^l, \tag{36}$$

where $u = (\gamma_2/\beta)^{1/2}x_0$, yields

$$\langle F(T)\rangle = \sum_{l=0}^{\infty}\frac{\Gamma(1/2)\Gamma(4+l/2)\zeta(4+l/2)}{\Gamma(4)\zeta(4)\Gamma(1/2+l/2)}\frac{\sin(l\pi/4)}{l\pi}u^l. \tag{37}$$

The leading terms in the series are

$$\langle F(T)\rangle = \frac{1}{4} + \frac{35\zeta(9/2)}{32\zeta(4)\sqrt{2}}u + \frac{4\zeta(5)}{\pi\zeta(4)}u^2 + \frac{105\zeta(11/2)}{64\zeta(4)\sqrt{2}}u^3 - \frac{693\zeta(13/2)}{256\zeta(4)\sqrt{2}}u^5 - \cdots. \tag{38}$$

This series converges for $|u|<1$.

V. ASYMPTOTIC SERIES

The power series in Eq. (37) can be analytically continued to give an asymptotic series in powers of $1/u$ for $|u|>1$ using a Barnes type of integral representation [2]. The power series is written here as

$$\langle F(T)\rangle = \frac{1}{4} + \frac{\Gamma(1/2)}{\Gamma(4)\zeta(4)}\,\mathrm{Im}\,J(u), \tag{39}$$

where

$$J(u) = \sum_{l=1}^{\infty}\frac{\Gamma(4+l/2)\zeta(4+l/2)}{\Gamma(1/2+l/2)l\pi}(ue^{i\pi/4})^l. \tag{40}$$

The continuation from the region $|u|<1$ to the region $|u|>1$ is mediated by the integral

$$I(u) = -\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} ds \frac{\Gamma(4+s/2)\zeta(4+s/2)}{\Gamma(1/2+s/2)s}\frac{(-ue^{i\pi/4})^s}{\sin \pi s}, \tag{41}$$

which with suitable restrictions is an analytic function of u, equals the sum in Eq. (40) for $|u|<1$, and can be expanded in an asymptotic series in powers of $1/u$ for $|u|>1$. This follows from consideration of various contour-integral representations of $J(u)$. The relevant contours are illustrated in Fig. 2, and are discussed in this section.

For s on the contour of integration in Eq. (41), the properties of the Γ and ζ functions yield [12]

$$\left|\frac{\Gamma(4+s/2)}{\Gamma(1/2+s/2)}\right| \to \left|\frac{s}{2}\right|^{7/2} \tag{42}$$

as $|s|\to\infty$ and

$$|\zeta(4+s/2)| \le \zeta(17/4). \tag{43}$$

Also with the definitions $-u=|u|e^{i\phi}$ and $s_2=\mathrm{Im}\,s$, we have for $s_2\to\pm\infty$

$$\left|\frac{(-ue^{i\pi/4})^s}{\sin \pi s}\right| \to 2|u|^{1/2}e^{-(\phi+\pi/4)s_2}e^{-\pi|s_2|}. \tag{44}$$

Evidently, the integral in Eq. (41) converges for ϕ in the range

$$|\phi+\pi/4| \le \pi-\delta, \tag{45}$$

where δ is a small positive number for any $|u|$.

For $|u|<1$, a sequence of integrals $I_N(u)$ is defined with the same integrand as in Eq. (41) but with different contours. Each modified contour is the perimeter of the area enclosed by the straight line $s=1/2+is_2$, where $-\infty<s_2<\infty$, and the nearly complete semicircle $s=(N+1/2)e^{i\theta}$, where N is an integer and $|\theta|<\pi/2$. In the limit $N\to\infty$, the integral of the portion along the straight line is just the integral in Eq. (41). On the circular contour, the estimates in Eqs. (42) and (43) are valid, whereas (see, for example, Ref. [12])

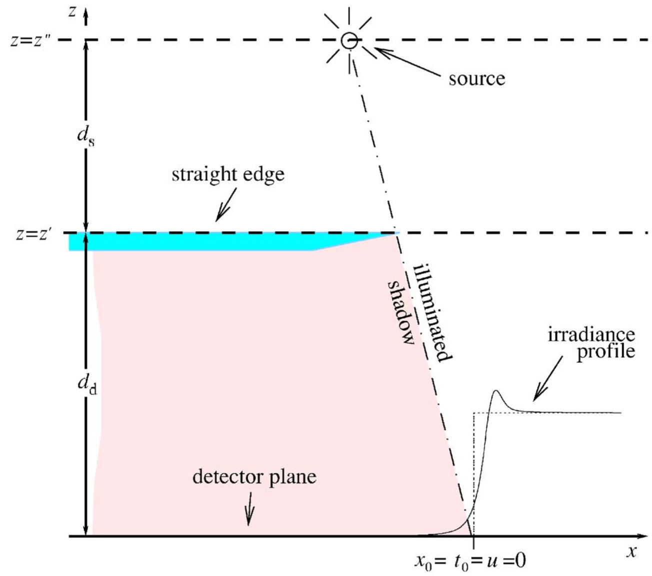

FIG. 1. (Color online) Diffraction of radiation from a small source by an edge. The dashed-dotted line indicates the shadow boundary, and the irradiance profile in a horizontal plane is shown for complex radiation. The dotted curve indicates the ideal, geometrical-optics irradiance profile.

$$\frac{(-ue^{i\pi/4})^s}{\sin \pi s}$$

$$= \begin{cases} O\{\exp[2^{-1/2}(N+1/2)\ln|u|]\} & \text{for } 0 \le |\theta| \le \pi/4 \\ O\{\exp[-2^{-1/2}(N+1/2)\delta]\} & \text{for } \pi/4 \le |\theta| \le \pi/2 \end{cases}. \tag{46}$$

Hence, the integral along the circular contour approaches zero as $N \to \infty$, provided $|u|<1$, and $I_N(u) \to I(u)$ in this limit.

For each integral $I_N(u)$, the contour is a closed path encircling (in the negative sense) N poles at the zeros of the sine function at positive integer values of s. The integral is then equal to $-2\pi i$ times the sum of the residues of these poles, which gives

$$I_N(u) = \sum_{l=1}^{N} \frac{\Gamma(4+l/2)\zeta(4+l/2)}{\Gamma(1/2+l/2)l\pi}(ue^{i\pi/4})^l, \tag{47}$$

so that $I_N(u) \to J(u)$ as $N \to \infty$, and consequently $I(u)=J(u)$ for $|u|<1$.

The integral $I(u)$ provides a convenient way to calculate the intensity for either $|u|<1$ or $|u|>1$ as a single integral. We have done this and the result is in agreement with the result of a direct calculation of the Planck-Fresnel intensity based on conventional Fresnel integrals together with a numerical evaluation of Eq. (35). An earlier calculation of this quantity done differently is described in Ref. [13]. The Planck-Fresnel intensity profile is shown in Fig. 3.

To obtain the asymptotic expansion for $|u|>1$, the contour in Eq. (41) is deformed to a new contour composed of three contributions: C_K, C_R, and C_H giving integrals denoted by $K_M(u)$, $R_M(u)$, and $H_M(u)$, respectively. The contour C_K encircles the real axis from $-M$ to $1/2$ and back in the positive sense, C_R extends from $-M-iv$ to $-M+iv$, and C_H consists of two horizontal segments from $-M+iv$ to $1/2+iv$ and from $1/2-iv$ to $-M-iv$, where M is an odd positive integer and $v \to \infty$. In this limit, we have

$$I(u) = K_M(u) + R_M(u) + H_M(u). \tag{48}$$

For the function $H_M(u)$, on the horizontal contours C_H, the estimate in Eq. (42) applies to the ratio of gamma functions, the zeta function has the form [14]

$$\zeta(4+s/2) = O(|v|^p), \tag{49}$$

where p is a constant that depends on $-M$, i.e., the minimum value of Re s on the contour, and

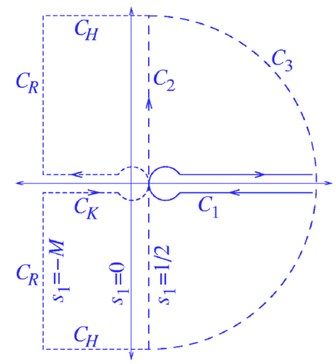

FIG. 2. (Color online) Contours related to the evaluation of $J(u)$ are shown. Contour C_1 defines $I(u)$ in Eq. (41). This contour can be replaced by the sum of contours C_2, the vertical axis with $s_1=1/2$, and C_3, the nearly complete semicircular contour in the right half plane. The contours C_H, C_R, and C_K are discussed in the text. The contributions of $H_M(u)$ and $R_M(u)$ are bounded, whereas $K_M(u)$ is evaluated term by term to obtain the final asymptotic expansion in Eq. (81).

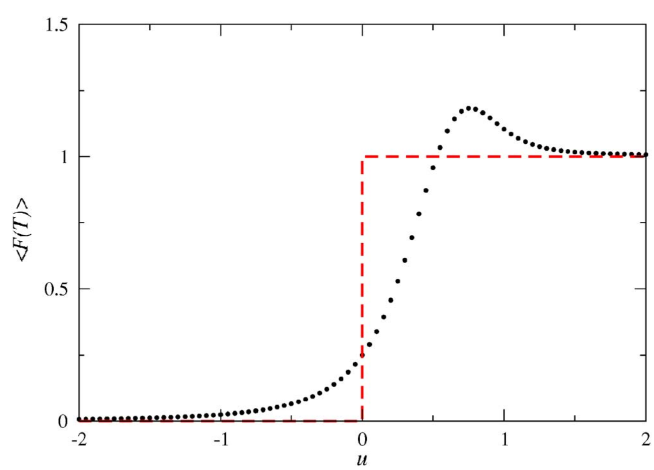

FIG. 3. (Color online) Straight-edge Fresnel diffraction effects on the irradiance profile for the case of a Planck source, as a function of $u=(\gamma_2/\beta)^{1/2}x_0$. Positive (negative) values of u correspond to the illuminated (shadow) portion of the detector plane. The dashed line shows what is expected from geometrical optics, whereas the points take diffraction into account.

$$\left|\frac{(-ue^{i\pi/4})^s}{\sin \pi s}\right| = O(e^{-\delta|v|}). \quad (50)$$

Thus, $H_M(u) \to 0$ as $|v| \to \infty$.

To evaluate the contribution from the vertical contour C_R, it is useful to take into account the Riemann relation

$$\Gamma(4+s/2)\zeta(4+s/2) = \frac{(2\pi)^{4+s/2}\zeta(-3-s/2)}{2\cos(\pi s/4)} \quad (51)$$

and the relation

$$\frac{1}{\Gamma(1/2+s/2)\sin \pi s} = \frac{\Gamma(1/2-s/2)}{2\pi \sin(\pi s/2)}, \quad (52)$$

which yield

$$R_M(u) = -\frac{1}{2\pi i}\int_{-M-iv}^{-M+iv} ds \frac{\Gamma(1/2-s/2)\zeta(-3-s/2)(2\pi)^{3+s/2}(-ue^{i\pi/4})^s}{2s\cos(\pi s/4)\sin(\pi s/2)}. \quad (53)$$

On the contour in Eq. (53) we have

$$|\zeta(-3-s/2)| \le \zeta(M/2-3) \le \zeta(3/2) \quad \text{for } M \ge 9, \quad (54)$$

$$|(2\pi)^{3+s/2}(-ue^{i\pi/4})^s| \le (2\pi)^3(\sqrt{2\pi}|u|)^{-M}e^{(\pi-\delta)|s_2|}, \quad (55)$$

$$|2\cos(\pi s/4)| = [2\cosh(\pi s_2/2)]^{1/2} > e^{\pi s_2/4}, \quad (56)$$

$$|\sin(\pi s/2)| = \cosh(\pi s_2/2) > e^{\pi s_2/2}/2, \quad (57)$$

so that as $v \to \infty$ and for $M \ge 9$, the remainder is bounded by

$$|R_M(u)| < \frac{2^{3/2}\pi^2\zeta(3/2)}{(\sqrt{2\pi}|u|)^M}(S_1+S_2), \quad (58)$$

where

$$S_1 = \int_0^M ds_2 \frac{|\Gamma(1/2+M/2+is_2)|}{|M+is_2|}e^{(\pi/4-\delta)s_2} < \frac{\Gamma(1/2+M/2)}{M}\frac{e^{(\pi/4-\delta)M}-1}{\pi/4-\delta}, \quad (59)$$

and

$$S_2 = \int_M^\infty ds_2 \frac{|\Gamma(1/2+M/2+is_2)|}{|M+is_2|}e^{(\pi/4-\delta)s_2}. \quad (60)$$

In the integrand in S_2, it is necessary to include the exponential damping of the gamma function for large imaginary argument. In this case $(s_2 > M)$, we write

$$|\Gamma(1/2+M/2+is_2/2)|^2 = \left(\frac{1}{2}\right)^M \left[\prod_{\substack{j=2\\ \text{even}}}^{M-1}(j^2+s_2^2)\right]\frac{\pi s_2}{\sinh(\pi s_2/2)} < \frac{\pi s_2^M}{2^{(M+1)/2}\sinh(\pi s_2/2)} \quad (61)$$

so that

$$S_2 < \frac{\pi^{1/2}}{2^{(M+3)/4}M}\int_M^\infty ds_2 \frac{s_2^{M/2}e^{(\pi/4-\delta)s_2}}{[\sinh(\pi s_2/2)]^{1/2}} < \frac{\pi^{1/2}}{2^{(M+1)/4}M}\int_0^\infty ds_2 \frac{s_2^{M/2}e^{-\delta s_2}}{(1-e^{-\pi M})^{1/2}} = \frac{\pi^{1/2}\Gamma(M/2)}{2^{(M+5)/4}\delta^{(M/2+1)}(1-e^{-\pi M})^{1/2}}. \quad (62)$$

Hence, $|R_M(u)|$ is bounded by $c_M/|u|^M$, where c_M is a constant independent of $|u|$ and is given by Eqs. (58), (59), and (62).

For the function $K_M(u)$, the integral is

$$K_M(u) = -\frac{1}{2\pi i}\int_{C_K} ds \frac{\Gamma(1/2-s/2)\zeta(-3-s/2)(2\pi)^{3+s/2}(-ue^{i\pi/4})^s}{2s\cos(\pi s/4)\sin(\pi s/2)}, \quad (63)$$

where the contour C_K is a closed path encircling (in the positive sense) the poles of the integrand on the real s axis between $-M$ and $1/2$, and the integral is evaluated by summing the residues of the poles inside C_K. The poles are at zero and the even negative integer values of s, so it is convenient to write the term arising from the singularity at $s=-2m$, where m is a non-negative integer, as

$$T_{-2m} = -\frac{1}{2\pi i}\int_{C_{-2m}} ds \frac{\Gamma(1/2-s/2)\zeta(-3-s/2)(2\pi)^{3+s/2}(-ue^{i\pi/4})^s}{2s\cos(\pi s/4)\sin(\pi s/2)}$$
$$= \frac{1}{2\pi i}\int_{C_0} dt \frac{\Gamma(t+m+1/2)\zeta(t+m-3)(2\pi)^3}{2(t+m)\cos[\pi(t+m)/2]\sin[\pi(t+m)](2\pi|u|^2 e^{i(2\phi+\pi/2)})^{t+m}}, \quad (64)$$

where C_{-2m} is a contour with a radius less than $1/2$ encircling the point $s=-2m$ in the positive sense, and the change of variable from s to t, where $s=-2t-2m$, is made to arrive at the second line. Some of the poles in the range of integration are of order two, so it is necessary to expand the relevant factors in the integrand to order t. The required expansions are as follows:

$$\Gamma(t+m+1/2) = \Gamma(m+1/2)[1+t\psi(m+1/2)+O(t^2)], \quad (65)$$

$$\zeta(t+m-3) = \begin{cases} \frac{1}{120}\left[1+t\left(\ln 2\pi+\gamma-\frac{11}{6}-\frac{\zeta'(4)}{\zeta(4)}\right)+O(t^2)\right] & m=0 \\ -\frac{\zeta(3)t}{4\pi^2}[1+O(t)] & m=1 \\ -\frac{1}{12}[1+O(t)] & m=2 \\ -\frac{1}{2}[1+t\ln 2\pi+O(t^2)] & m=3 \\ \frac{1}{t}[1+t\gamma+O(t^2)] & m=4 \\ \zeta(m-3)\left[1+t\frac{\zeta'(m-3)}{\zeta(m-3)}+O(t^2)\right] & m\geq 5, \end{cases} \quad (66)$$

$$\frac{1}{t+m} = \begin{cases} \frac{1}{t} & m=0 \\ \frac{1}{m}\left[1-\frac{t}{m}+O(t^2)\right] & m\neq 0, \end{cases} \quad (67)$$

$$\frac{1}{\cos[\pi(t+m)/2]} = \begin{cases} i^m[1+O(t^2)] & m \text{ even} \\ \frac{i^{m+1}2}{\pi t}[1+O(t^2)] & m \text{ odd}, \end{cases} \quad (68)$$

$$\frac{1}{\sin[\pi(t+m)]} = \frac{(-1)^m}{\pi t}[1+O(t^2)], \quad (69)$$

$$\frac{1}{(2\pi|u|^2 e^{i(2\phi+\pi/2)})^{t+m}} = \left(\frac{-i}{2\pi u^2}\right)^m [1-t(\ln 2\pi|u|^2+i2\phi + i\pi/2)+O(t^2)]. \quad (70)$$

In Eq. (65), ψ is the logarithmic derivative of the gamma function, and in Eq. (66), γ is Euler's constant. Evaluation of the residues yields

$$T_0 = -\frac{\pi^{5/2}}{30}\left[\frac{11}{6}+\frac{\zeta'(4)}{\zeta(4)}+\ln 4|u|^2+i\left(2\phi+\frac{\pi}{2}\right)\right], \quad (71)$$

$$T_{-2} = \frac{i\zeta(3)}{2\pi^{3/2}u^2}, \quad (72)$$

$$T_{-4} = -\frac{\pi^{1/2}}{32u^4}, \quad (73)$$

$$T_{-6} = \frac{i5}{16\pi^{3/2}u^6}\left[\frac{41}{15}-\gamma-\ln 4|u|^2-i\left(2\phi+\frac{\pi}{2}\right)\right], \quad (74)$$

$$T_{-8} = \frac{105}{256\pi^{3/2}u^8}\left[\frac{1303}{420}-\ln 8\pi|u|^2-i\left(2\phi+\frac{\pi}{2}\right)\right], \quad (75)$$

$$T_{-2m} = -\frac{i8\pi\Gamma(m+1/2)\zeta(m-3)}{m(2\pi u^2)^m}\left[\psi(m+1/2)+\frac{\zeta'(m-3)}{\zeta(m-3)} - \frac{1}{m}-\ln 2\pi|u|^2-i\left(2\phi+\frac{\pi}{2}\right)\right], \quad m\geq 5 \text{ odd}, \quad (76)$$

$$T_{-2m} = \frac{4\pi^2\Gamma(m+1/2)\zeta(m-3)}{m(2\pi u^2)^m}, \quad m\geq 6 \text{ even}. \quad (77)$$

The terms with $m=2$ and $m=3$ are special cases of the general expressions in Eqs. (77) and (76), whereas the terms with $m=0$, $m=1$, and $m=4$ are not. These terms are exceptional because for $m=0$, there is an extra factor of t^{-1} in Eq. (67), for $m=1$, the zeta function has a zero, and for $m=4$, the zeta function has a pole. We thus have the asymptotic expansion

$$I(u) = \sum_{m=0}^{(M-1)/2} T_{-2m} + O(|u|^{-M}). \quad (78)$$

For the intensity profile, only terms with an imaginary part contribute, i.e., those with $m=0$, $m=1$, $m=3$, $m=4$, and odd $m \geq 5$. A further simplification is that u is real in the expression for the intensity profile, so that from the definition of ϕ, which was given by $-u=|u|e^{i\phi}$, and the restriction on the range of ϕ in Eq. (45), ϕ is given by

$$\phi = -\theta(u)\pi. \quad (79)$$

The result for the asymptotic expansion of the intensity profile is

$$\langle F(T)\rangle = \frac{1}{4} + \frac{\Gamma(1/2)}{\Gamma(4)\zeta(4)} \sum_{m=0}^{(M-1)/2} \mathrm{Im}\, T_{-2m} + O(|u|^{-M}), \quad (80)$$

or

$$\langle F(T)\rangle = \theta(u) + \frac{15\zeta(3)}{2\pi^5 u^2} + \frac{75}{16\pi^5 u^6}\left[\frac{41}{15} - \gamma - \ln 4u^2\right] + \frac{1575}{128\pi^4 u^8}\left[\theta(u) - \frac{1}{4}\right] - \sum_{\substack{m=5\\ \text{odd}}}^{N} \frac{8\pi\Gamma(1/2)\Gamma(m+1/2)\zeta(m-3)}{\Gamma(4)\zeta(4)m(2\pi u^2)^m}\left[\psi(m+1/2) + \frac{\zeta'(m-3)}{\zeta(m-3)} - \frac{1}{m} - \ln 2\pi u^2\right] + O(|u|^{-2N-3}). \quad (81)$$

VI. CONCLUSION

The Fresnel intensity distribution has been obtained as a power series in powers of the distance from the geometrical shadow boundary by solving a differential equation for that quantity. The result is given by Eq. (34). Term-by-term integration of this series gives the intensity pattern for the Planck distribution of frequencies as a power series in Eq. (38). This power series is analytically continued by means of a Barnes integral representation to give the asymptotic series for $|u| > 1$ in Eq. (81).

We conclude with the following observations. First, evaluation of two terms of the asymptotic expansion

$$\langle F(T)\rangle_2 = \theta(u) + \frac{15\zeta(3)}{2\pi^5 u^2} \quad (82)$$

provides a good numerical approximation to the intensity profile for $|u| \geq 1$ except near $u=1$, as shown by the numbers in Table I. Also, it is noteworthy that the intensity profile, which for a single wavelength involves the squares of Fresnel integrals, can be written as a powers series with simple coefficients in which every fourth term (after the zeroth term) is zero. Further, this power series has a contour integral representation amenable to a Barnes-type evaluation, giving rise to the asymptotic expansion for a Planck source. This asymptotic expansion has the property that except for the leading geometrical shadow contribution and one term proportional to u^{-8}, all terms are symmetric with respect to the geometrical shadow boundary and are proportional to $u^{-(4i+2)}$, where i is a non-negative integer.

TABLE I. Comparison of the truncated exact numerical values and the values given by the two-term asymptotic expansion in Eq. (82) for the Planck-Fresnel intensity. The exact values are obtained by direct numerical integration of Eqs. (15) and (35).

u	$\langle F(T)\rangle$	$\langle F(T)\rangle_2$	$\langle F(T)\rangle - \langle F(T)\rangle_2$
−8	0.000 460	0.000 460	0.000 000
−4	0.001 833	0.001 841	−0.000 008
−2	0.007 122	0.007 365	−0.000 143
−1	0.024 696	0.029 460	−0.004 764
1	1.103 369	1.029 460	0.073 910
2	1.007 616	1.007 365	0.000 251
4	1.001 835	1.001 841	−0.000 006
8	1.000 460	1.000 460	0.000 000

[1] M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge University Press, Cambridge, 1999).
[2] E. W. Barnes, Proc. London Math. Soc. **6**, 141 (1907).
[3] C. L. Sanders and O. C. Jones, J. Opt. Soc. Am. **52**, 731 (1962).
[4] N. Ooba, J. Opt. Soc. Am. **54**, 357 (1964).
[5] W. R. Blevin, Metrologia **6**, 39 (1970).
[6] L. P. Boivin, Appl. Opt. **14**, 2002 (1975).
[7] L. P. Boivin, Appl. Opt. **16**, 377 (1977).
[8] E. L. Shirley, Appl. Opt. **43**, 2609 (2004).
[9] E. L. Shirley, J. Opt. Soc. Am. A **21**, 1895 (2004).
[10] G. Lawrence, G. Kopp, G. Rottman, J. Harder, T. Woods, and H. Loui, Metrologia **40**, S78 (2003).
[11] R. W. Brusa and C. Fröhlich, Appl. Opt. **25**, 4173 (1986).
[12] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed. (Cambridge University Press, Cambridge, 1927).
[13] E. L. Shirley, Appl. Opt. **43**, 2609 (2004).
[14] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, 2nd ed. (Clarendon Press, Oxford, 1986).